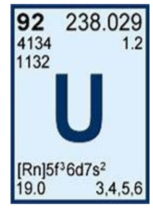
EXPONENTIAL EQUATIONS – NO LOGS

Watching your investments grow, tracking populations, the decaying of a radioactive substance — these are the kinds of problems that lead to exponential equations.



Uranium, atomic number 92, decays exponentially – the less that remains, the less that decays.

❑ *EXAMPLES*

An ***exponential equation*** is an equation with the variable in the exponent, something you might not be used to seeing. Let's start with two basic examples.

#1: Consider the exponential equation

$$5^x = 25$$

Five raised to what power equals 25? Well, 5 to the **2nd** power is 25, so $\boldsymbol{x = 2}$. Not so tough.

#2: How about the equation

$$3^{2n-1} = 81?$$

You might solve this equation by asking yourself, "3 to what power is 81?" Since 3 to the 4th power is 81, it follows that the exponent, $2n - 1$, must be equal to 4.

Writing this last phrase as an equation, we can find the value of n:

$$2n - 1 = 4 \;\Rightarrow\; 2n = 5 \;\Rightarrow\; \boldsymbol{n = \frac{5}{2}}. \text{ Done!}$$

Check: Placing $n = \frac{5}{2}$ into the equation $3^{2n-1} = 81$ gives

$$3^{2\left(\frac{5}{2}\right)-1} \stackrel{?}{=} 81$$

$$3^{\cancel{2}\left(\frac{5}{\cancel{2}}\right)-1} \stackrel{?}{=} 81$$

$$3^{5-1} \stackrel{?}{=} 81$$

$$3^{4} \stackrel{?}{=} 81$$

$$81 = 81 \checkmark$$

EXAMPLE 1: **Solve for x: $3^{7x} = 3^{14}$**

Solution: Each side of the equation is an exponential expression. Notice that the bases are the same (the 3's), so the only way the two sides of the equation can be equal is *if the exponents are equal*. In other words, $7x$ must equal 14:

$$7x = 14$$

from which we determine that

$$\boxed{x = 2}$$

EXAMPLE 2: **Solve for y: $5^{4y} = 25$**

Solution: We're not as lucky here as we were in Example 1 — the bases are not the same. But maybe we can make them the same. Suppose we think of 25 as 5^2. Then, each side of the equation will have the same base, and we can set the exponents equal to each other to find the value of y. Let's try all of this:

$$5^{4y} = 25 \qquad \text{(the original equation)}$$

$$\Rightarrow \quad 5^{4y} = 5^2 \qquad \text{(rewrite 25 with a base of 5)}$$

$$\Rightarrow \quad 4y = 2 \qquad \text{(set the exponents equal to each other, since the bases are the same)}$$

$$\Rightarrow \quad \boxed{y = \frac{1}{2}}$$

EXAMPLE 3: **Solve for z: $27^{-4z} = \frac{1}{9}$**

Solution: This equation's terrible! The bases aren't the same: One is 27 and the other is 1/9. And it contains a fraction. But look at the 27 in the equation — it equals 3^3. And check out the 9 in the denominator — it can be written as 3^2. So maybe we can write everything in terms of the base **3**:

$$27^{-4z} = \frac{1}{9} \qquad \text{(the original equation)}$$

$$\Rightarrow \quad (3^3)^{-4z} = \frac{1}{3^2} \qquad \text{(express 27 and 9 as powers of 3)}$$

$$\Rightarrow \quad 3^{-12z} = 3^{-2} \qquad \text{(exponent rules)}$$

$$\Rightarrow \quad -12z = -2 \qquad \text{(bases are the same – set exponents equal)}$$

$$\Rightarrow \quad \boxed{z = \frac{1}{6}} \qquad \text{(solve for } z\text{)}$$

Homework

Solve each exponential equation:

1. $16^{6p} = 8$
2. $4^{10q} = 16$
3. $125^{3t} = \frac{1}{5}$
4. $16^{6w} = 16$
5. $4^{-2h} = 64$
6. $25^{6b} = \frac{1}{25}$
7. $25^{8w} = 125$
8. $625^{-4x} = \frac{1}{625}$
9. $125^{-10a} = \frac{1}{25}$
10. $16^{3v} = 2$
11. $4^{9r} = \frac{1}{2}$
12. $9^{-4p} = 729$
13. $625^{-3y} = \frac{1}{25}$
14. $81^{9x} = \frac{1}{729}$
15. $25^{-2k} = \frac{1}{25}$
16. $16^{-5x} = 512$
17. $27^{8b} = \frac{1}{729}$
18. $81^{6a} = 3$
19. $81^{-5r} = 81$
20. $625^{6b} = \frac{1}{25}$
21. $27^{-5v} = 3$
22. $81^{-4k} = \frac{1}{9}$
23. $125^{5p} = \frac{1}{25}$
24. $125^{-10d} = \frac{1}{5}$

<u>EXAMPLE 4</u>: **Solve each equation for x:**

A. $e^{5x} = e^{20}$

$\Rightarrow\ 5x = 20 \ \Rightarrow\ \boldsymbol{x = 4}$

B. $e^{2x+1} = (e^3)^{x-5}$

$\Rightarrow\ e^{2x+1} = e^{3x-15} \ \Rightarrow\ 2x+1 = 3x-15$

$\Rightarrow\ -x = -16 \ \Rightarrow\ \boldsymbol{x = 16}$

C. $\frac{1}{e} = e^{2x-1}$

$$\Rightarrow e^{-1} = e^{2x-1} \Rightarrow -1 = 2x-1 \Rightarrow \boldsymbol{x = 0}$$

D. $\frac{1}{\sqrt[3]{e}} = \left(\frac{1}{e^2}\right)^x$

$$\Rightarrow \frac{1}{e^{1/3}} = \frac{1^x}{(e^2)^x} \Rightarrow \frac{1}{e^{1/3}} = \frac{1}{e^{2x}}$$

$$\Rightarrow \frac{1}{3} = 2x \Rightarrow \boldsymbol{x = \frac{1}{6}}$$

<u>EXAMPLE 5:</u> **Solve for x: $3^x = 5$**

<u>Solution:</u> Now we're really up a stump! How can we make the bases the same? We can't, so our current method for solving exponential equations dies at this point. In fact, the solution to this equation will not be a rational number as in the previous four examples, so the best we can do is approximate it. Here's one way to solve the problem, using a calculator (without using a logarithm key).

This is simply a guess-and-check method. Clearly, the solution x of the equation $3^x = 5$ is bigger than 1, since $3^1 = 3$; but x must be smaller than 2, since $3^2 = 9$. So the solution is between 1 and 2:

$$1 < x < 2$$

Now try 1.5. $3^{1.5} = 5.196$, a little too big. Let $x = 1.4$; then $3^{1.4} = 4.656$, a little too small. Thus, we've narrowed the solution to a number between 1.4 and 1.5:

$$1.4 < x < 1.5$$

Since 1.45 is halfway between 1.4 and 1.5, let's give it a try. $3^{1.45} = 4.918$, which is really close to 5. Consider 1.46: $3^{1.46} = 4.973$. Now nudge x up to 1.47: $3^{1.47} = 5.028$, a little too big. So the value of x is between 1.46 and 1.47:

$$1.46 < x < 1.47$$

This is getting boring, but I hope you're getting the point. Let's just take the average of 1.46 and 1.47, and figure we're close enough.

Our best guess is therefore

$$\boxed{x = 1.465}$$

In fact, if we check our "solution," we get $3^{1.465} = 5.0001$, so we have a really accurate value of x. The following chart is a summary of the calculations we made in this problem:

x	1	1.4	1.45	1.46	1.465	1.47	1.5	2
3^x	3	4.656	4.918	4.973	5.0001	5.028	5.196	9

In a future chapter we will study what are called "log" functions. Then we'll have a third (and much better) way of solving any kind of exponential equation, even if the base is e.

Homework

Solve each equation for exact solutions:

25. $2^{7x+1} = 2^{3x}$
26. $6^{8y-1} = 36$
27. $8^{-6z} = \frac{1}{2}$
28. $\frac{1}{25^x} = 125^{3x-1}$
29. $7^{3x} = 7^{5x}$
30. $81^{4-3x} = \sqrt{729}$
31. $e^{2x+1} = e$
32. $\frac{1}{e^2} = e^{1-7x}$
33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e}$

Use your calculator to approximate the solution of each equation:

34. $2^x = 6$ 35. $10^y = 50$ 36. $e^z = 12$

37. Solve by inspection (this means stare at it and guess): $8^x = 5^x$

Solutions

1. $p = \frac{1}{8}$ **2.** $q = \frac{1}{5}$ **3.** $t = -\frac{1}{9}$ **4.** $w = \frac{1}{6}$

5. $h = -\frac{3}{2}$ **6.** $b = -\frac{1}{6}$ **7.** $w = \frac{3}{16}$ **8.** $x = \frac{1}{4}$

9. $a = \frac{1}{15}$ **10.** $v = \frac{1}{12}$ **11.** $r = -\frac{1}{18}$ **12.** $p = -\frac{3}{4}$

13. $y = \frac{1}{6}$ **14.** $x = -\frac{1}{6}$ **15.** $k = \frac{1}{2}$ **16.** $x = -\frac{9}{20}$

17. $b = -\frac{1}{4}$ **18.** $a = \frac{1}{24}$ **19.** $r = -\frac{1}{5}$ **20.** $b = -\frac{1}{12}$

21. $v = -\frac{1}{15}$ **22.** $k = \frac{1}{8}$ **23.** $p = -\frac{2}{15}$ **24.** $d = \frac{1}{30}$

25. $-\frac{1}{4}$ **26.** $6^{8y-1} = 6^2 \Rightarrow 8y - 1 = 2 \Rightarrow y = \frac{3}{8}$

27. $8^{-6z} = \frac{1}{2} \Rightarrow (2^3)^{-6z} = 2^{-1} \Rightarrow 2^{-18z} = 2^{-1}$

$\Rightarrow -18z = -1 \Rightarrow z = \frac{1}{18}$

28. $\frac{3}{11}$ **29.** 0 **30.** $\frac{13}{12}$ **31.** 0 **32.** $\frac{3}{7}$

33. $\left(\frac{1}{\sqrt{e}}\right)^x = \sqrt[4]{e} \Rightarrow \left(e^{-1/2}\right)^x = e^{1/4} \Rightarrow e^{-\frac{1}{2}x} = e^{\frac{1}{4}}$

$\Rightarrow -\frac{1}{2}x = \frac{1}{4} \Rightarrow x = -\frac{1}{2}$

34. 2.58 **35.** 1.7 **36.** 2.48

37. Hint: The solution is the only number with no reciprocal.

"*Education* is not merely a means for earning a living or an instrument for the acquisition of wealth. It is an initiation into life of spirit, a training of the human soul in the pursuit of truth, and the practice of virtue."

– Vijaya Lakshmi Pandit